Math Circles - Intro to Combinatorics - Winter 2024

Solution Set 1

February 7th, 2024

1. How many licence plates that are seven characters long, without repetition can you make using only letters and digits? What if we require a letter as the first character?
Solution: There are $10 + 26 = 36$ total options for each character. Thus $36 \times 35 \times 34 \times 33 \times 32 \times 31 \times 30 = \frac{36!}{29!}$ options. This is also $P(36, 7)$. If we require the first character to be a letter, then we have $26 \times 35 \times 34 \times 33 \times 32 \times 31 \times 30 = 26 \times \frac{35!}{29!} = 26P(35, 6)$.

2. Suppose you are a wedding photographer. If a wedding party has 6 people, including the bride and groom, then how many ways can we arrange the photo if

   (a) the bride is next to the groom?
   Solution: Consider the bride and groom as one object. Then observe we are arranging 5 objects. So 5! ways are available. For each way we can arrange the bride and groom in two ways. Thus we have $2 \times 5!$ total arrangements.

   (b) the bride is not next to the groom?
   Solution: There are 6! ways to arrange all the people and in (a) we observed there were $2 \times 5!$ ways to arrange the people with the bride and groom next to each other. Thus $6! - 2 \times 5!$ arrangements exist with the bride not next to the groom.

   (c) the bride is to the left, but not necessarily next to, the groom?
   Solution: Given any arrangement the bride and groom can be swapped to create another arrangement. This also swaps if the bride is to the left or right of the groom. So the bride is to the left in exactly half the arrangements. Thus $\frac{6!}{2}$ total ways exist to place the bride to the left of the groom. Note there are other ways to obtains this answer.

3. How many ways can a group of group of four people be chosen from a group of 8 adults and 5 children is the groups must have two adults and two children?
Solution: There are $\binom{8}{2}$ ways to pick the adults and $5]choose2$ ways to pick the children. Since we can pick independently and want the total number of combinations we use the product rule to multiply the options. Hence we have $\binom{8}{2}\binom{5}{2}$ total ways to pick the group.

4. How many different card hands in poker are a flush? A deck of cards has 52 cards, 4 suits, and a flush is a set of five cards all of the same suit.
Solution: There are four suits and each has an equal number of cards. Thus there are 13 cards in a suit. So given a suit we have $\binom{13}{5}$ flushes. Thus we have $\binom{4}{1}\binom{13}{5}$ flushes. Another way to see this is for each suit we have $\binom{13}{5}$ flushes and we can add these together by the sum rule, giving $\binom{13}{5} + \binom{13}{5} + \binom{13}{5} + \binom{13}{5} = 4\binom{13}{5}$. Note $\binom{4}{1} = 4$, so our two answers are equal.

5. How many solutions, using natural numbers (positive numbers) and zero are there to the equation $x + y + z = 20$?
Solution: We use the trick of placing dividers. We can imagine 20 as a list of 20 ones. We will place two dividers into the ones and x will be the number of ones before the first divider, y the number of ones between the first and second divider, and z the number after the second divider. So we have 22 spots to place the dividers and we want to place 2. Thus $\binom{12}{2}$ total solutions.

6. How many words would you have to write, in any language using the 26 letter alphabet we use in English, before you have to have written two words that have the same first and last letter?
Solution: There are 26×26 first and last letter combinations. Since we don't care about the middle of the word we can write $26 \times 26 = 676$ words before we must repeat a combination. Thus after writing 677 words we have to have repeated a combinations.